ZX-Rules for 2-qubit Clifford+T Quantum Circuits

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Abstract

We give the ZX-rules that enable one to derive all equations between 2-qubit Clifford+T circuits. Our rule set only extends stabilizer ZX-calculus with a single new rule, and hence is substantially less than those needed for the recently achieved universal completeness. In particular, these ZX-rules are much simpler than the complete of set Clifford+T circuit equations due to Selinger and Bian. The underlying reason is that ZX-calculus is not constrained by a fixed unitary gate set for performing intermediate computations.

Moreover, recently it was shown by Renaud Vilmart that our new rule suffices for universal completeness of ZX-calculus, the only difference between 2-qubit Clifford+T circuit rules and universal ZX-diagram rules being the variables to be used within the rule: in the case of universal completeness all explicit phase values are needed, while in the 2-qubit case one never needs to consider explicit phases, only relationships between these.

We also conjecture that while in that sense ZX-rules scale with increasing number of qubits, circuit rules don't, and propose a pathway to a prove thereof.

The 2-qubit axioms

Theorem. *The rules (S1), (S2), (B1), (B2), (H1), (H2), (N) and (P) depicted below make*

ZX-calculus complete for 2-qubit Clifford+T circuits:
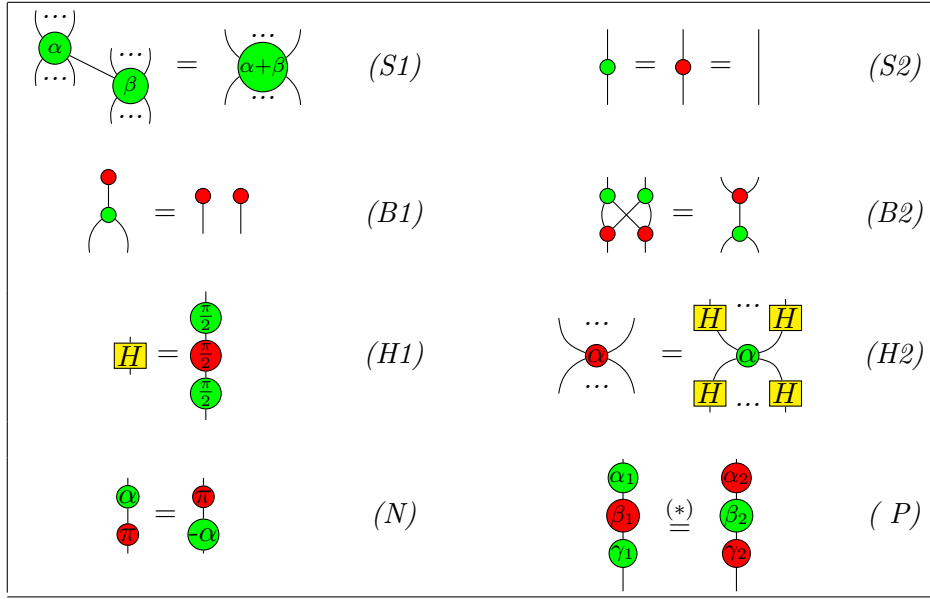
where $\alpha_2 = \gamma_2$ *if* $\alpha_1 = \gamma_1$, *and* $\alpha_2 = \pi + \gamma_2$ *if* $\alpha_1 = -\gamma_1$; *the equality (*) should be read as follows: for every diagram in LHS there exists* α_2, β_2 *and* γ_2 *such that LHS=RHS (and vice versa if conjugating by the Hadamard gate). In what follows we will see that we actually don't need to know the precise values of* α_2, β_2 *and* γ_2.

In particular, these rules are much simpler, intuitive, and memorisable than the corresponding circuit relations as for example established by Selinger-Bian.

Universal Completeness

To achieve (scalar-free) universal completeness, the (P) rule above should be extended to all values. This is indeed a key difference between the 2-qubit Clifford+T, where the actual values in the (P) rule are never needed; only some special relationships between these are needed. This general case of the (P) rule was also derived in our paper:

Proposition. *We have:*

 (1)

with $\alpha_1, \beta_1, \gamma_1 \in (0, \ 2\pi), \alpha_2 = \arg z + \arg z_1, \gamma_2 = \arg z - \arg z_1, \beta_2 = 2\arg(|\frac{z}{z_1}| + i)$, *where* $z = \cos\frac{\beta_1}{2}\cos\frac{\alpha_1+\gamma_1}{2} + i\sin\frac{\beta_1}{2}\cos\frac{\alpha_1-\gamma_1}{2}$, $z_1 = \cos\frac{\beta_1}{2}\sin\frac{\alpha_1+\gamma_1}{2} - i\sin\frac{\beta_1}{2}\sin\frac{\alpha_1-\gamma_1}{2}$, $z_2 = |\frac{z}{z_1}| + i$. *If* $\alpha_1 = \gamma_1$, *then* $\alpha_2 = \gamma_2$; *If* $\alpha_1 = -\gamma_1$, *then* $\alpha_2 = \pi + \gamma_2 (Mod\ 2\pi)$.

The paper moreover also contains a further generalisation of this result to the generalised phases employed in the Ng-Wang ZX universal completeness theorem.

A Conjecture on scalability

From the above it in particular follows that, in a certain sense, the ZX-rules for qubit circuits scale to circuits of an arbitrary number of qubits. We expect that the same is not at all true for circuit relations such as those by Selinger-Bian. We conjecture that the following 3-qubit circuit, which represents the Frobenius law, cannot be derived from them:
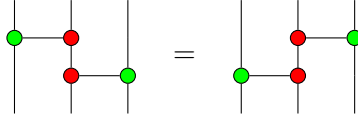
$$\vcenter{\hbox{\includegraphics{circuit1}}} \quad = \quad \vcenter{\hbox{\includegraphics{circuit2}}}$$

nor can it be derived from any 2-qubit relations. So there are two possible strategies for addressing the ZX vs. circuits scalability issue:

- do so for the specific Selinger-Bian rules, or,

- do so for any set of two-qubit rules.

We tried the 1st one, came really close, but no cigar (yet). The latter is a stronger result, but also avoids having to mess with the specific ugliness of the S-B rules.